*Proceedings of the International Association for Shell and Spatial Structures (IASS)*
*Symposium 2015, Amsterdam*
*Future Visions*
17 - 20 August 2015, Amsterdam, The Netherlands

# Design and exploration of externally post-tensioned structures using graphic statics

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## Abstract

Funicular structures, which follow the shapes of hanging chains, work in pure tension (cables) or pure compression (arches), and offer a materially efficient solution compared to structures that work through bending action. However, the set of geometries that are funicular under common loading conditions is limited. Non-structural design criteria, such as function, program, and aesthetics, often prohibit the selection of purely funicular shapes, resulting in large bending moments and excess material usage. In response to this issue, this paper explores the use of a new design approach that converts non-funicular planar curves into funicular shapes without changing the geometry; instead, funicularity is achieved through the introduction of new loads using external post-tensioning. The methodology is based on graphic statics, and is generalized for any two-dimensional shape. The problem is indeterminate, meaning that a large range of allowable solutions is possible for one initial geometry. Each solution within this range results in different internal force distributions and horizontal reactions. The method has been implemented in an interactive parametric design environment, empowering fast exploration of diverse axial-only solutions. In addition to presenting the approach and tool, this paper provides a series of case studies and numerical comparisons between new post-tensioned structures and classical bending solutions, demonstrating that significant material can be saved without compromising on geometrical requirements.

**Keywords**: equilibrium; graphic statics; structural design; post-tensioning system; geometry; funicular structures; design methods

# 1. Introduction

## 1.1. Motivation

In the design of curved structures, the overall shape, usually decided in conceptual design, is a key contributor to the overall cost and structural efficiency. Non-structural conditions, such as aesthetics, functionality, and geotechnical issues, often prohibit selection of a structurally ideal funicular shape in which only axial forces are activated (the word funicular is independently used for only compression or tension curves). In contrast to a hanging chain, forces may not be able to act in pure compression or tension under self-weight, and bending moments inevitably arise, increasing the structural material required.

In form-finding problems, funicular curves can be found from a given set of loads. This paper explores the opposite problem: starting from a known geometry, and its related distribution of dead loads, how it is possible to find a system of external loads that, in combination with the existing loads, can convert the starting geometry into a funicular curve.

## 1.2. Historical Background

This paper applies the concept of funicularity [1-4] to curves that are originally not funicular under dead loads, in order to take advantage of the structural efficiency due to a funicular behavior while also allowing for architectural and geometric flexibility. A classical design approach for non-funicular structures is based on the use of bending solutions, in which axial forces and bending coexist, or trusses. In contrast, this paper rather explores the possibility of adding external loads on a two-dimensional curve for changing the internal forces distributions, and consequently for improving its structural behavior. An existing example of this concept is represented by the façade of the Pavilion of the Future built in Seville for the Universal Exposition in 1992, designed by Peter Rice from Ove Arup & Partners. The main design concept takes advantage of the gravity loads generated by roof beams to apply a set of radial equal forces onto the circular arch of the façade, forcing the thrust line to pass closer to the center of the arch [5].

Figure 1. Façade of the Pavilion of the Future (1992). Images adapted from Lenczner [5].

Inspired by this project, the research presented in this paper uses a generalized approach in which external forces are applied to the initial non-funicular curve through an external post-tensioning system of stressed tension cables connected to the non-funicular geometry with compressive/tension struts. External post-tensioning systems can improve design performance significantly by converting arbitrary curves into geometries that better resist permanent loads with axial forces only and can be implemented to many structural typologies, empowering the designer to control the structural behavior of curved structures [6]. This presented design philosophy has been generalized by Todisco et al. [7] through an approach based on graphic statics.

This paper mainly presents step-by-step the theoretical framework behind this approach, showing how this concept can be applied to designs in a broad sense. Furthermore, taking advantage of an implemented design-oriented tool, several examples showing how the method can be used on a variety of practical design problems are illustrated. Finally, the paper also presents results in the form of more detailed evaluations of the system, using simplified numerical comparisons of the results with the classical bending solution.

## 2. Theoretical framework

### 2.1. Graphical construction

The new approach presented in this paper is based on graphic statics, which relates structural geometry and internal forces through dual polygons. It is graphical in the sense that forces are calculated geometrically from the force polygon. In this approach, equilibrium equations are taken into account, and it is a very powerful method for designing new statically determinate structures and for analyzing existing ones. Exhaustive literature on this topic can be found in textbooks [1, 8-11] or into other recent research publications [12-15]. A similar graphical approach, limited to find the geometry of trusses with constant chord force, has been presented by Lachauer [16]. The developed methodology here is divided in two main steps: first, starting from a given shape, a set of loads is found in order to make the curve funicular; second, a post-tensioning tendon layout for generating these loads is found. The graphical approach is detailed in the following:

a) The graphical construction starts with a given planar geometry. In the illustrated example of Figure 2, it is a circular one (Fig. 2a), but the methodology can be applied to any two-dimensional curve. The starting geometry is loaded with a known distribution of dead loads.

b) The curve has to be subdivided into a polyline, and every subdivision can be represented by a vertical load vector, whose size depends on the corresponding dead load (Fig. 2b). This hypothesis represents the first limit of the method: it has been developed for a discrete number of loads. The higher the number of subdivisions, the lower the discrepancy with the real behavior of the structure. As shown in Fig. 2b, because of the shape is not funicular for its dead load, the force diagram is not closed, and equilibrium with only axial forces is not possible.

c) On each vertex, the direction of the strut carrying external loads has to be chosen. In Fig. 2c every strut is assumed to be normal to the curve, although the procedure is not limited to radial loads. The direction of the struts has to relate to the construction process and to its effects on the tension curve. The graphical construction for closing the form diagram is shown in Fig. 2c where new forces, combined with the existing ones, are needed at each vertex in order to close the force polygon. Just as infinite funicular geometries exists for any single distribution of loads, an infinite set of loads exists for making a single geometry funicular. In other words, the direction of the internal forces is fixed but their magnitude is variable. This assumption represents the first indeterminacy of the problem. The solution is unique if one parameter is fixed, i.e. the horizontal reaction or the compressive force in the curve.

d) Fig. 2d shows another equilibrated solution obtained with a different magnitude of loads. In this case the forces introduced in each vertex are higher.

e) Once one distribution of external forces inducing a state of only compressive stresses has been identified, the graphical construction for finding the external post-tensioning shape generating the forces of Fig.2d must be performed. The start and end points of the tension curve should be defined. In the described examples, these points coincide with the supports in order to evaluate their contribution to the horizontal reactions. The force polygon can be graphically constructed, guaranteeing the equilibrium at each joint of the structure. First an arbitrary pole is selected, and then lines connecting this point to the force vector can be drawn, as shown in Fig. 2e. The position of the pole is arbitrary because to one set of loads corresponds infinite cable layout in order to generate that load distribution. This is the second indeterminacy of the problem. Fixing one parameter, i.e. the maximum value of post-tensioning force or defining the starting inclination of the tendon, yields a unique solution; there is a unique relationship between post-tensioning force and the geometry of the external system. The solution shown in Fig. 2e is a special case because all the forces in the tension curve have the same magnitude (constant post-tensioning force) and this allows for an optimization of the cable.

f) Fig. 2f shows another equilibrated solution for generating the required loads and to make the starting curve funicular. In this case the pole has been moved and this corresponds to different axial forces in the cable.

g) Fig. 2g shows the complete graphical construction obtained superimposing Figs. 2d and 2e.

h) Fig. 2h shows the complete graphical construction obtained superimposing Figs. 2d and 2f.

i) Fig. 2i illustrates the horizontal and vertical components of the reactions obtained from the graphical construction of Fig. 2g. In this case, the post-tensioning cable starts with a vertical inclination and it has no contribution to the horizontal reaction.

l) Fig. 2j shows the reactions obtained from the graphical construction of Fig. 2h. Different from the previous case shown in Fig. 2i, the tension cable increases the horizontal reaction. If the cable had an inward inclination, the horizontal reaction would be lower.

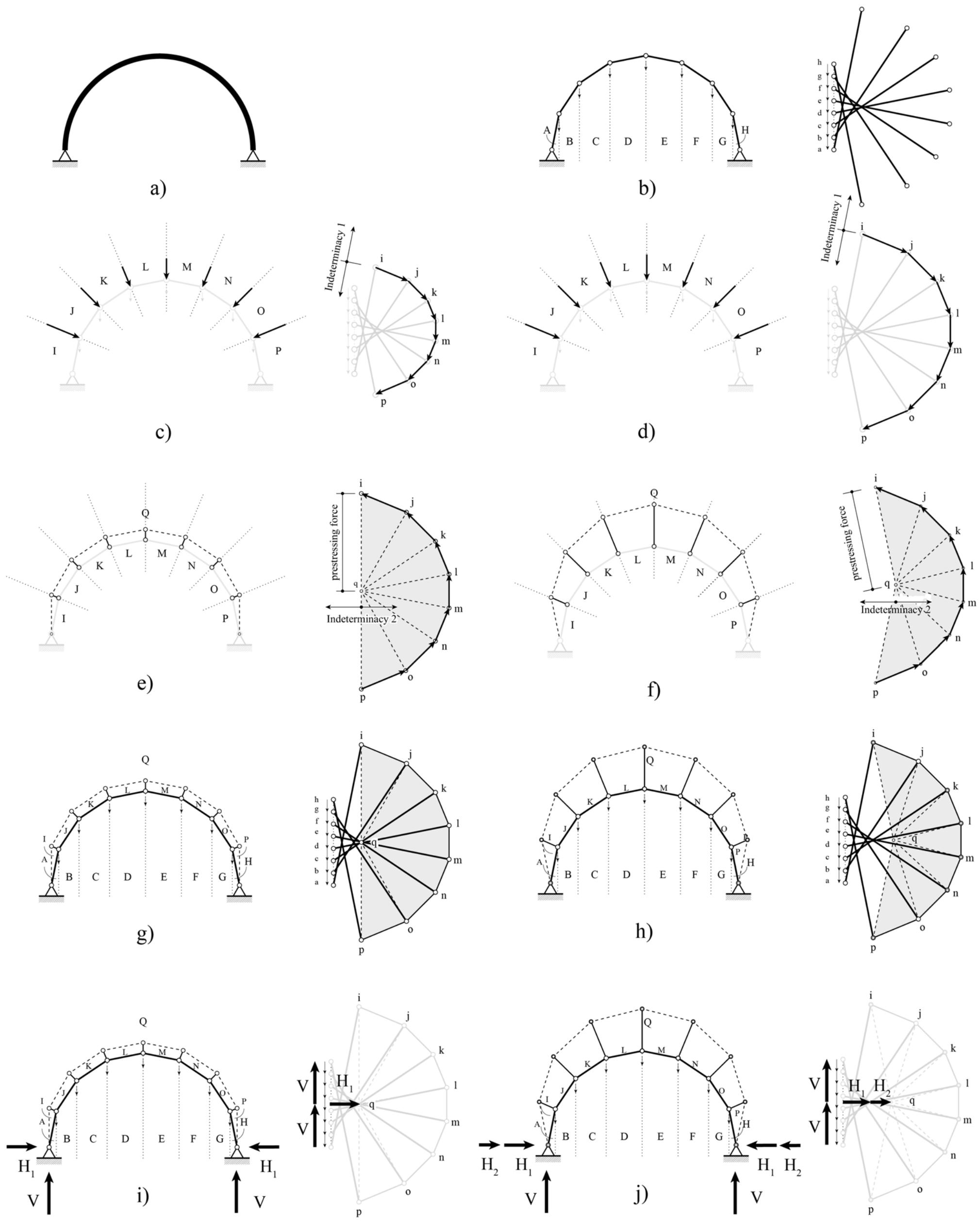


Figure 2. Graphical construction of the external post-tensioning system.

## 2.2. Examples of application

The use of graphic statics for design, as opposed to analysis, has been limited until recently, in part because a variation of the geometry or loads changes the model and the graphical construction, which must be updated, is time-consuming to complete manually. In order to overcome this limitation, a new interactive tool for finding the layout of external post-tensioning systems has been developed. It is interactive in that a change of the geometry of the starting shape results automatically in a new tendon layout in static equilibrium. Equilibrium is always ensured by the dual reciprocity between the force diagram and form polygon. A more in-depth description of the tool is reported in Todisco et al. [7]. As mentioned previously, a similar interactive tool for finding the geometry of trusses using graphic statics has been already developed by Lachauer [16].

The presented tool has been implemented using the parametric plug-in Grasshopper [17] for the software Rhinoceros [18] and it will be available for free download and use from the authors' websites. For the definition of the starting model four inputs are needed: two-dimensional shape, number of segment in which the curve should be divided, magnitude of dead loads and direction of the struts connecting the starting geometry with the post-tensioning system. The user taking advantage of this tool can explore the infinite range of equilibrated solutions for any starting geometry.

Figure 3 shows three different design alternatives for an ellipsoidal arch. Changing the position of the pole related to the post-tensioning allows for a variation in support reactions. The third configuration shows how the inward horizontal thrust $H_1$ due to the dead load is equilibrated completely by the outward horizontal thrust $H_2$ due to the introduction of the post tensioning, generating no horizontal reactions at the supports. Furthermore, Figure 3 illustrates that with the same starting geometry, depending on the relative position between the cable and compressive curve, struts can carry compression or tension forces. This aspect can be relevant from the point of view of construction and connection design.

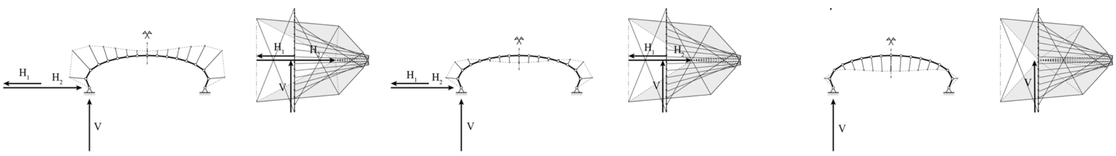

Figure 3. Examples of resulting post-tensioned forms for arbitrary geometries.

Figure 4 illustrates the application of the methodology to shapes in which the starting curve is far from funicular under dead load. The collection of examples, with varying degrees of curvature and both symmetrical and asymmetrical forms, demonstrates the versatility of the methodology and of the implemented tool for any kind of two-dimensional geometry. Forces in all structural members can be read in the force diagram, giving to the designer the possibility of evaluating the performance of the whole system consisting of compressive and post-tensioned curves, and the connecting struts. A visual comparison between magnitude of dead loads and post-tensioning forces gives an idea of how far is the starting geometry from being funicular for its dead loads. Figure 4 shows only one solution for each geometry, but, as described in Section 2, there are infinite possible shapes of the cable for each starting curve.

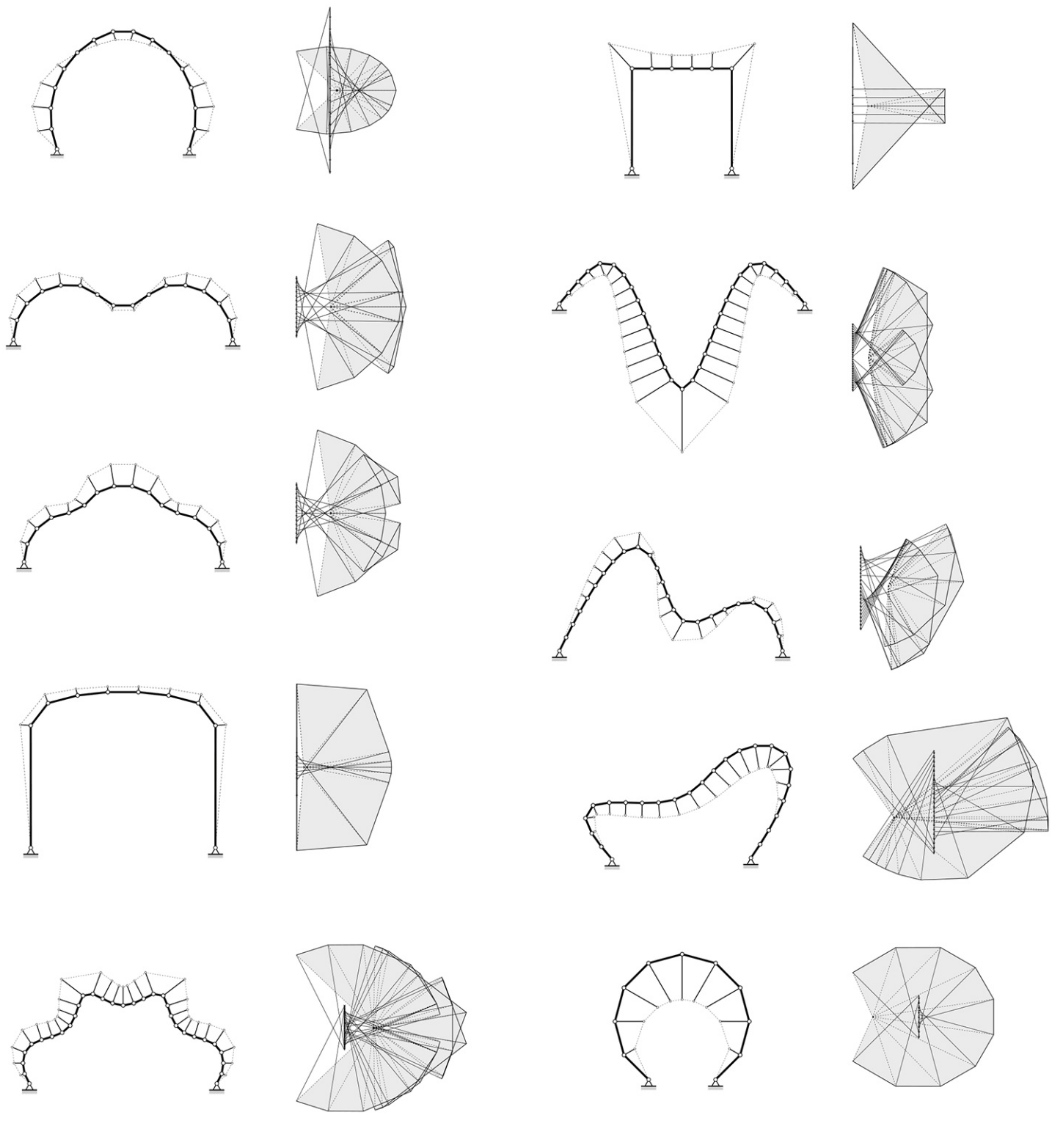

Figure 4. Examples of resulting post-tensioned forms for arbitrary geometries.

Results obtained by graphical construction have been further validated through reduced-scale non-funicular geometries fabricated through additive manufacturing (3D printing) [19] and a pavilion made from custom-cut corrugated cardboard and nylon webbing [20]. Physical models show that once the post-tensioning force is applied, the structures, made of discrete elements, are stable.

## 3. Exploration and evaluation of design alternatives

### 3.1. Case studies

While the interpretation of the force polygon reveals the magnitude of forces in the system, more detailed evaluations must be performed in order to find limitations of the described design philosophy. In order to evaluate the effects on the global structural behavior of obtaining a funicular geometry using an external post-tensioning system, a simplified comparison between bending and funicular solutions has been performed for three non-funicular geometries.

The starting data are the geometry of the initial curve (all of them have a span of 30 m), the direction of the struts (radial) and the dead load magnitude (12 kN/m). Concrete circular hollow profiles (thickness/diameter=0.05, $f_y$=30 N/mm2) have been adopted for the compressive curve. The cross section design, computed within the elastic range, depends on the forces in the structural elements, and the minimum section satisfying both global buckling of the structure and yield stresses is taken.

The result consequent to the application of the previous described post-tensioning method for a known geometry is not determinate, i.e. there are infinite solutions for each starting geometry. The user can explore the infinite range of equilibrated solutions by varying related parameters, as the horizontal reactions, the magnitude of axial forces in the compressive curve and the post-tensioning force applied to the tendon.

The external forces to be added to the starting geometry, to make it funicular, are related to the maximum allowable offset between the compressive curve and the post-tensioned tendon. This distance between the two curves is described by a percentage of the span and it is a relevant parameter for evaluating the efficiency of the system. As a general rule, the smaller the distance between the compressive curve and the post-tensioning system, the higher the forces in both components.

### 3.2. Analysis of results

The design of the bending solution, which considers combined axial forces and bending moments, is governed by the allowable stress and requires an hollow cross section which total mass for the non-funicular structure is M*. The design of the funicular solution, which considers only axial forces (higher compared to the bending solution), is governed by allowable stress and requires a hollow cross section; the total mass of the compressive curve is M.

Figure 5 shows the comparison in term of required mass of the starting geometry (M/M*) for, respectively, three-pointed arch, circular arch and free-form asymmetrical curve. Each computed comparison is related to the maximum offset between post-tensioning system and compressive curve. Defined a target maximum offset, the minimum set of forces to be added to the existing loads have been selected for the comparison.

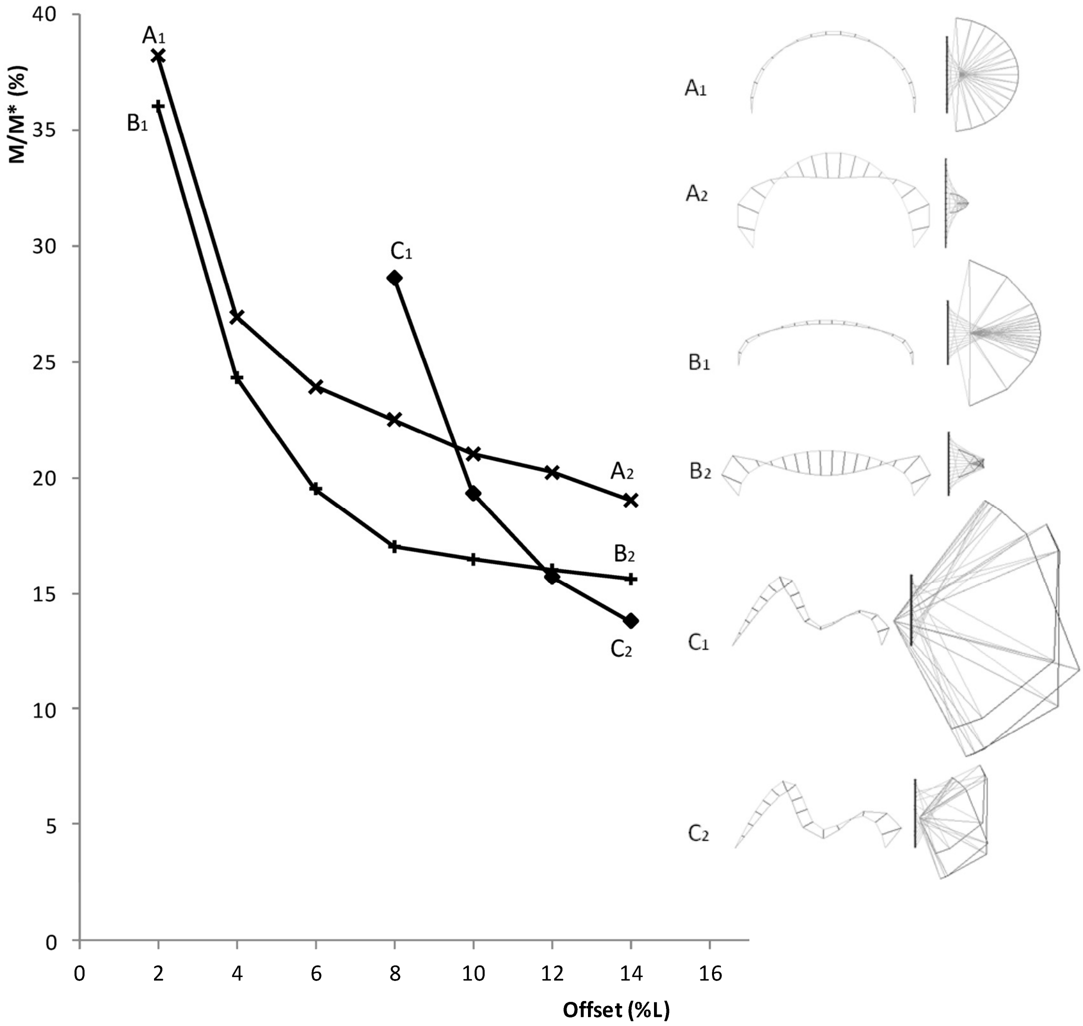


Figure 5. Comparison between bending and post-tensioned solution for a three-pointed arch, a semi-circular arch and a free-form curve with different offsets.

Results obtained for different offsets illustrate how the position of the post-tensioned tendon strongly influences the structural performance for all the analyzed shapes. If the cable has a minimum offset (i.e. it is closed to the compressive curve) high values of external forces are needed, compressive forces in the curved geometry are higher, inevitably decreasing the efficiency of the proposed system. Furthermore, the reading of the force polygons reveals the difference between dead loads and forces to be added for making the geometry funicular and shows how far a shape is from being funicular for its loads. In this example, the asymmetrical shape results the one with highest forces to be introduced.

However, independent of the offset, the overall structural behavior of the post-tensioned solution results in savings of 60% to 85% of the whole mass compared with the bending solution. Results illustrate that despite the introduction of higher axial forces, the act of removing bending moments allows to design more cost-effective structures.

While this simplified methodology is not sufficient for a complete design, the approach is consistent with the aim of the analysis: a first simplified comparison between different solutions for a preliminary evaluation of their structural performances.

## 4. Conclusions and future work

The choice of an appropriate funicular shape in curved geometries is an important decision for designing materially efficient structures. However, optimal shapes from a structural point of view are often at odds with functionality, and non-funicular geometries must be used.

The goal of this paper is to address this problem by introducing a new method for designing external post-tensioning systems that add forces to the non-funicular geometries, transforming the distribution of loads and making them act in pure compression. The presented approach for finding the geometry of the post-tensioning system is based on a graphical method. It has been implemented in an interactive design-oriented tool that allows users to explore new design alternatives for a given two-dimensional geometry. The combination of the graphical-based design technique embedded in an interactive environment gives great freedom in designing new shapes for buildings and spatial structures. Simplified numerical comparisons show how post-tensioning allows almost any shape to be funicular, improving the overall structural behavior and reducing the required mass compared to the bending solution in which axial and bending coexist.

In this paper, the comparison between bending and post-tensioned solution has been performed only for one material (concrete) and cross-section typology (hollow circular profiles). More parametric analysis is needed for evaluating the advantages of adopting a post-tensioned system instead of a classical bending or trussed solution. In these analyses asymmetrical loads, different materials, various starting geometries should be take into account. Not only the mass, but the global cost of the structure (including post-tensioning system and connecting struts) should be computed for a more precise performance evaluation of the whole system. Finally, construction aspects should be taken into account in the choice of post-tensioning layout and the direction of struts.

## Acknowledgement

This work has been done through a collaborative agreement between the Polytechnic University of Madrid, Spain and the Massachusetts Institute of Technology, US. The first author wish to express him gratitude for the financial support received from the Chair of Structural Concrete of the Polytechnic University of Madrid. The authors want to sincerely thank Juney Lee from MIT, for greatly improving the design examples and graphics in this paper. The authors are grateful to John

Ochsendorf from the Massachusetts Institute of Technology for all the valuable information and support provided.

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